Economic Theory **17**, 675–692 (2001)

Economic Theory

for economies with single-dipped preferences and the assignment of an indivisible object*

Bettina Klaus

Department of Economics, University of Nebraska at Lincoln, Lincoln, NE 68588-0489, USA
(e-mail: bklaus1@unl.edu)

Received: September 29, 1999; revised version: March 22, 2000

Summary. We study two allocation models. In the first model, we consider the problem of allocating an infinitely divisible commodity among agents with single-dipped preferences. In the second model, a degenerate case of the first one, we study the allocation of an indivisible object to a group of agents.

We consider rules that satisfy *Pareto efficiency*, *strategy-proofness*, and in addition either the consistency property *separability* or the solidarity property *population-monotonicity*.

We show that the class of rules that satisfy *Pareto efficiency*, *strategy-proofness*, and *separability* equals the class of rules that satisfy *Pareto efficiency*, *strategy-proofness*, and *non-bossiness*. We also provide characterizations of all rules satisfying *Pareto efficiency*, *strategy-proofness*, and either *separability* or *population-monotonicity*. Since any such rule consists for the largest part of serial-dictatorship components, we can interpret the characterizations as impossibility results.

Keywords and Phrases: Strategy-proofness, Serial-dictatorship, Population-monotonicity, Separability, Non-bossiness.

JEL Classification Numbers: D63, D71.

* Part of this paper is a revision of the working paper "Compatibilities and Incompatibilities in Economies with Single-Dipped Preferences". I wish to thank William Thomson and a referee for many helpful comments and Youngsub Chun for bringing the property of *separability* to my attention.

1 Introduction

Consider the allocation of a perfectly divisible commodity among a group of agents with so-called “single-dipped” preferences: preferences are single-dipped if alternatives can be ordered in such a way that every agent has a single worst alternative, his dip amount, and his welfare strictly increases in either direction away from his dip amount. For example, consider two types of work which have negative cross-effects such as teaching and administration in a university: combinations of the two types of work have lower utility than pure one-type tasks. Yet other examples are two-good exchange economies with fixed prices and strictly quasi-convex utility functions, or exchange economies with classical economic preferences but with “nonstandard”, kinked budget curves (see Klaus, 1998, Example 1.3); in both cases preferences, when restricted to the budget curves, are single-dipped.

Although the domain of single-dipped preferences is less well-known than, for instance, the domain of single-peaked preferences (e.g., Chun, 1998a; Klaus 1998; Thomson 1994, 1995a), there are interesting economic situations with underlying single-dipped preferences. In a public good context Vickrey (1960) refers to single-dipped preferences as “single-troughed”. Inada (1964) studies single-dipped preferences over triples of alternatives.[1] Peremans and Storcken (1997) consider the problem of locating a public facility with strongly negative externalities. These externalities induce single-dipped preferences on the set of admissible locations.

For the problem of allocating an infinitely divisible commodity among agents with monotonic or single-peaked preferences a large class of rules satisfying normatively appealing properties exist, e.g., the equal division rule for monotonic preferences or the so called uniform rule for single-peaked preferences (Chun, 1998a; Klaus, 1998; Thomson, 1994, 1995a). In the case of single-dipped preferences, many combinations of properties force a rule to assign the whole amount of the commodity to a single agent in a (sequentially) dictatorial way. This paper pursues an axiomatic study of rules for economies with single-dipped preferences. In two earlier papers (Klaus, Peters, and Storcken, 1997; Klaus, 1999) we started studying the trade-off between properties of rules for economies with such preferences. Two central properties in this analysis are *Pareto efficiency* and *strategy-proofness*. A rule is *strategy-proof* if no agent can ever gain by misrepresenting his preferences, irrespective of the preferences announced by the other agents. Both properties are compatible for the model at hand, but the class of *Pareto efficient* and *strategy-proof* rules is large. This allows us to impose further requirements on the class of rules. Klaus, Peters, and Storcken (1997) and Klaus (1999) add several fairness properties (e.g., *no-envy*), *replacement-domination*, *consistency*, *weak non-bossiness in terms of welfare*, and *coalitional strategy-proofness* to the basic properties. In this paper we focus on *separability*, *non-bossiness*, and *population-monotonicity*.

[1] In Inada (1964) single-dipped preferences are called single-caved.

When comparing two economies that are defined over the same set of agents, *separability* requires the following. If each agent in a subgroup has the same preference relation in both economies and the total amount assigned to this subgroup is the same in both economies, then the amounts assigned to each agent in the subgroup should be the same in both economies. A rule is *non-bossy* if an agent cannot influence some other agent's allotment without affecting his own allotment. The solidarity property *population-monotonicity* requires that after the arrival of new agents, either all agents initially present (weakly) gain together or they all (weakly) lose together.

First we characterize the class of rules that satisfy *Pareto efficiency*, *strategy-proofness*, and either *non-bossiness* or *separability*. Any such rule must allocate the whole social endowment to a single agent. This assignment follows a procedure that is partly serially dictatorial. Next, we study the impact of *population-monotonicity* on *Pareto efficient* and *strategy-proof* rules: the whole social endowment must be assigned to a single agent. Again, any rule that satisfies all the required properties can be decomposed into serially-dictatorial components. Klaus, Peters, and Storcken (1997) and Klaus (1999) provide similar characterizations involving the solidarity property *replacement-domination*, the incentive properties *coalitional strategy-proofness* and *(weak) non-bossiness in terms of welfare*, and *consistency*. We conclude the allocation model for the perfectly divisible commodity with a brief discussion of other properties (*converse consistency* and *resource-monotonicity*).

Finally, we discuss the problem of assigning one indivisible object to a group of agents. This problem in itself would be interesting enough to study, but as we show it is closely connected to the problem of allocating an infinitely divisible commodity among agents with single-dipped preferences. Because agents can only report their preferences over the alternatives, "receiving the object" and "not receiving the object", *Pareto efficiency* implies *strategy-proofness*. Similarly as before, we can characterize the class of rules that satisfy *Pareto efficiency* and either *non-bossiness* or *population-monotonicity.*

2 Preliminaries

The model

There is a finite population of potential agents, indexed by $\mathbb{P}$. By $\mathscr{P}$ we denote the class of non-empty subsets of $\mathbb{P}$. Each agent $i \in N$ is equipped with a single-dipped preference relation R_i defined over the non-negative real numbers $\mathbb{R}_+$.[2] Single-dippedness of R_i means that there exists a point $d(R_i) \in \mathbb{R}_+$, the *dip amount of agent* i, with the following property: for all $x, y \in \mathbb{R}_+$ with $x < y \leq d(R_i)$ or $x > y \geq d(R_i)$ we have $x \, P_i \, y$.[3] By $\mathscr{D}$ we denote the class of all

[2] Without loss of generality, we may assume that the agents' preference relations are continuous as well.

[3] As usual, $x \, R_i \, y$ is interpreted as "x is weakly preferred to y", and $x \, P_i \, y$ as "x is strictly preferred to y". Furthermore, $x \, I_i \, y$ means that agent i is indifferent between x and y.

single-dipped preference relations over $\mathbb{R}_+$. For $N \in \mathscr{P}$, $\mathscr{D}^N$ denotes the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathscr{D}$.

Let $\Omega \in \mathbb{R}_+$ be the *(social) endowment* that has to be distributed among a group of agents $N \in \mathscr{P}$ with profile $R \in \mathscr{D}^N$. Now, an *economy* is a pair $e = (R, \Omega) \in \mathscr{D}^N \times \mathbb{R}_+$. Let $\mathscr{E}^N := \mathscr{D}^N \times \mathbb{R}_+$ and $\mathscr{E} := \bigcup_{N \in \mathscr{P}} \mathscr{E}^N$ be the class of all economies. A *feasible allocation for* $e = (R, \Omega) \in \mathscr{E}^N$ is a vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = \Omega$. An *allocation rule* φ, or a *rule* for short, is a function that assigns to every $e \in \mathscr{E}$ a feasible allocation, denoted $\varphi(e)$. Given $i \in N$, we call $\varphi_i(R)$ the *allotment* of agent i.

Pareto efficiency and strategy-proofness

We are interested in rules that select *Pareto efficient* allocations: such an allocation cannot be changed in a way that makes no agent worse off and some agent better off.

Pareto efficiency: For all $N \in \mathscr{P}$ and all $e = (R, \Omega) \in \mathscr{E}^N$, there is no feasible allocation $x \in \mathbb{R}^N_+$ such that for all $i \in N$, $x_i \, R_i \, \varphi_i(e)$, and for some $j \in N$, $x_j \, P_j \, \varphi_j(e)$.

First, we present a simple description of *Pareto efficiency* for our model. For that purpose we "partition" the set of the agents. For each economy $e = (R, \Omega) \in \mathscr{E}^N$ denote the set of agents who strictly prefer Ω to 0 by $N_\Omega(e) = \{i \in N \mid \Omega \, P_i \, 0\}$, the set of agents who are indifferent between 0 and Ω by $N_{0,\Omega}(e) = \{i \in N \mid 0 \, I_i \, \Omega\}$, and the set of agents who strictly prefer 0 to Ω by $N_0(e) = \{i \in N \mid 0 \, P_i \, \Omega\}$. Hence, for all $e = (R, \Omega) \in \mathscr{E}^N$, $N = N_\Omega(e) \cup N_{0,\Omega}(e) \cup N_0(e)$ and the sets $N_\Omega(e)$, $N_{0,\Omega}(e)$, and $N_0(e)$ are pairwise disjoint.[4]

Lemma 1 (Efficiency Lemma) *A rule φ is Pareto efficient if and only if for all $N \in \mathscr{P}$ and all $e = (R, \Omega) \in \mathscr{E}^N$ the following holds:*

Case 1 : *If $N_\Omega(e) \neq \emptyset$, then*
for all $i \notin N_\Omega(e)$, $\varphi_i(e) = 0$ and
for all $i \in N_\Omega(e)$, either $\varphi_i(e) = 0$ or $\varphi_i(e) \, P_i \, 0$.[5]

Case 2 : *If $N_\Omega(e) = \emptyset$ and $N_{0,\Omega}(e) \neq \emptyset$, then*
for some $j \in N_{0,\Omega}(e)$, $\varphi_j(e) = \Omega$.

Case 3 : *If $N_0(e) = N$, then*
for all $i \in N$, either $\varphi_i(e) = \Omega$ or $\varphi_i(e) \, P_i \, \Omega$.[6]

Proof. See Klaus (1999), proof of Lemma 1. □

[4] However, since some of the sets $N_\Omega(e)$, $N_{0,\Omega}(e)$, and $N_0(e)$ are possibly empty, strictly speaking, $\{N_\Omega(e), N_{0,\Omega}(e), N_0(e)\}$ may not constitute a partition.

[5] See Figure 1 a.

[6] See Figure 1 b.

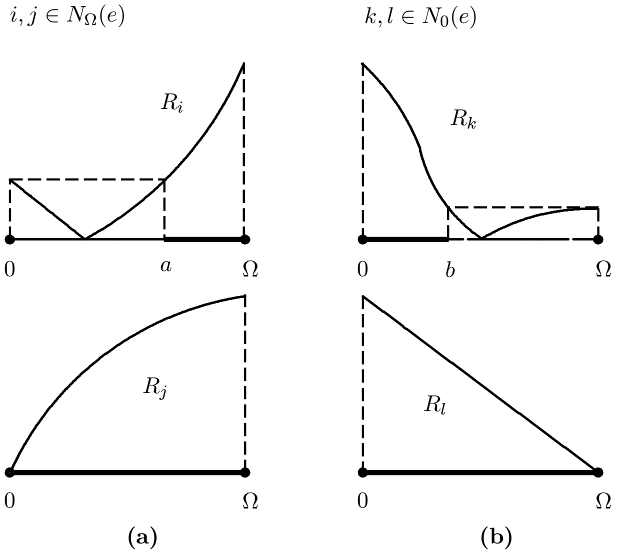

Figure 1a,b. *Pareto efficient* allotments as described in the Efficiency Lemma. **a** Efficiency Lemma, Case 1: $N_\Omega(e) \neq \emptyset$. Necessary conditions for *Pareto efficiency* of allotments for agents $i, j \in N_\Omega(e)$, x_i, x_j, are: $x_i \in \{0\} \cup (a, \Omega]$ and $x_j \in [0, \Omega]$. **b** Efficiency Lemma, Case 3: $N_0(e) = N$. Necessary conditions for *Pareto efficiency* of allotments for agents $k, l \in N_0(e)$, y_k, y_l, are: $y_k \in [0, b) \cup \{\Omega\}$ and $y_l \in [0, \Omega]$

Next, in addition to *Pareto efficiency*, we are interested in *strategy-proofness*. *Strategy-proofness* states that no agent can ever benefit from misrepresenting his preferences.[7] Before we give the formal definition, we introduce some notation.

For $e = (R, \Omega) \in \mathscr{E}^N$, $i \in N$, and $\overline{R}_i \in \mathscr{D}$, $(\overline{R}_i, R_{-i})$ denotes the profile obtained from R by replacing R_i by $\overline{R}_i$. We call $\overline{R} = (\overline{R}_i, R_{-i})$ an *i-deviation from R*.

Strategy-proofness: For all $N \in \mathscr{P}$, all $e = (R, \Omega), \overline{e} = (\overline{R}, \Omega) \in \mathscr{E}^N$, and all $j \in N$, if $\overline{R}$ is a j-deviation from R, then $\varphi_j(e) R_j \varphi_j(\overline{e})$.

A strengthening of *strategy-proofness* is the following condition of *coalitional strategy-proofness*: no group of agents can ever benefit from misrepresenting their preferences.

Let $M \subseteq N$. For $R \in \mathscr{D}^N$ the *restriction* $(R_i)_{i \in M} \in \mathscr{D}^M$ *of R to M* is denoted by R_M. Let $C \subseteq N$. Then, $N \backslash C = \{i \in N \mid i \notin C\}$.

Coalitional strategy-proofness: For all $N \in \mathscr{P}$, all $e = (R, \Omega) \in \mathscr{E}^N$, and all $C \subseteq N$ there exists no $\overline{e} = (\overline{R}, \Omega) \in \mathscr{E}^N$ with $\overline{R}_{N \backslash C} = R_{N \backslash C}$ such that for all $i \in C$, $\varphi_i(\overline{e}) R_i \varphi_i(e)$ and for some $j \in C$, $\varphi_j(\overline{e}) P_j \varphi_j(e)$.

[7] In game theoretical terms, a rule is *strategy-proof* if in its associated direct revelation game form, it is a weakly dominant strategy for each agent to announce his true preference relation.

The following result will be useful later on; if there are exactly two agents, then a *Pareto efficient* and *strategy-proof* rule φ assigns the whole endowment to a single agent.

Let $\mathscr{B}^2_\varphi = \{e = (R, \Omega) \in \mathscr{E}^N \mid N \in \mathscr{P}, |N| = 2$, and for all $i \in N$, $\varphi_i(e) \neq \Omega\}$ be the set of two-agent economies that yield a *broken allocation* under φ, *i.e.*, a feasible allocation where none of the agents obtains the whole endowment.

Lemma 2 *Let φ be a rule that satisfies Pareto efficiency and strategy-proofness. Then, $\mathscr{B}^2_\varphi = \emptyset$.*

Proof. Let φ satisfy the properties listed in the lemma and suppose, by contradiction, that $\mathscr{B}^2_\varphi \neq \emptyset$. Then, there exists (R, Ω) such that $\varphi(R, \Omega)$ is broken. Since φ is *Pareto efficient* we can apply the Efficiency Lemma. Since the broken allocations in the description of the Efficiency Lemma can only occur in Cases 1 or 3, it follows that one of the following two cases holds.

(a) $N_\Omega(R, \Omega) = N = \{i, j\}$. Then, $\varphi_i(R, \Omega)\, P_i\, 0$ and $\varphi_j(R, \Omega)\, P_j\, 0$. Consider an i-deviation R^1 from R and a j-deviation R^2 from R^1 such that $\Omega\, P^1_i\, 0\, P^1_i\, \varphi_i(R, \Omega)$ and $\Omega\, P^2_j\, 0\, P^2_j\, \varphi_j(R, \Omega)$.

Strategy-proofness for the i-deviation R^1 from R implies $\varphi_i(R^1, \Omega) \leq \varphi_i(R, \Omega)$. By $N_\Omega(R^1, \Omega) = N_\Omega(R, \Omega)$ and the Efficiency Lemma, it follows that either $\varphi_i(R^1, \Omega) = 0$ or $\varphi_i(R^1, \Omega)\, P^1_i\, 0$. The latter can only be true if $\varphi_i(R^1, \Omega) > \varphi_i(R, \Omega)$, which contradicts *strategy-proofness*. Hence, $\varphi_i(R^1, \Omega) = 0$ and $\varphi_j(R^1, \Omega) = \Omega$. By *strategy-proofness* for the j-deviation R^2 from R^1, we have $\varphi_j(R^2, \Omega) = \Omega$. Similarly, by interchanging the roles of i and j, it follows that $\varphi_i(R^2, \Omega) = \Omega$, which is a contradiction.

(b) $N_0(R, \Omega) = N = \{i, j\}$. Then, $\varphi_i(R, \Omega)\, P_i\, \Omega$ and $\varphi_j(R, \Omega)\, P_j\, \Omega$. Consider an i-deviation R^1 from R and a j-deviation R^2 from R^1 such that $0\, P^1_i\, \Omega P^1_i\, \varphi_i(R, \Omega)$ and $0\, P^2_j\, \Omega\, P^2_j\, \varphi_j(R, \Omega)$.

Strategy-proofness for the i-deviation R^1 from R implies $\varphi_i(R^1, \Omega) \geq \varphi_i(R, \Omega)$. By $N_0(R^1, \Omega) = N$ and the Efficiency Lemma, it follows that either $\varphi_i(R^1, \Omega) = \Omega$ or $\varphi_i(R^1, \Omega)\, P^1_i\, \Omega$. The latter can only be true if $\varphi_i(R^1, \Omega) < \varphi_i(R, \Omega)$, which contradicts *strategy-proofness*. Hence, $\varphi_i(R^1, \Omega) = \Omega$. By *strategy-proofness* for the j-deviation R^2 from R^1, we have $\varphi_i(R^2, \Omega) = \Omega$. Similarly, by interchanging the roles of i and j, it follows that $\varphi_j(R^2, \Omega) = \Omega$, which is a contradiction. □

As we will see later, the result of Lemma 2 does not hold for economies with more than two agents (see Example 4).

3 Separability and non-bossiness

Our next goal is to characterize the class of rules that satisfy *Pareto efficiency*, *strategy-proofness*, and either one of two additional properties: *separability* or *non-bossiness*. Before defining *separability*, we introduce the stronger property of *consistency*. *Consistency* states that if a group of agents leaves a given economy with their allotments, then the allotments of the remaining agents are not redistributed by the rule in the new, “reduced” economy. For a recent survey of the literature on *consistency* we refer to Thomson (1996).

Let $M, N \in \mathscr{P}$ be such that $M \subsetneq N$. For $x \in \mathbb{R}^N$ the *restriction* $(x_i)_{i \in M} \in \mathbb{R}^M$ *of* x *to* M is denoted by x_M.

Consistency: For all $M, N \in \mathscr{P}$ with $M \subsetneq N$ and all $e = (R, \Omega) \in \mathscr{E}^N$, $\varphi(e)_M = \varphi(R_M, \sum_M \varphi_i(e))$.

For a further discussion of *consistency* for this model we refer to Klaus, Peters, and Storcken (1997). Here, we study a weaker consistency property called *separability*.

It requires that if for two economies with the same set of agents each agent in a subgroup has the same preference relation in both economies and the total amount assigned to this subgroup is the same in both economies, then the amounts assigned to each agent in the subgroup should be the same in both economies.

Separability: For all $M, N \in \mathscr{P}$ with $M \subsetneq N$ and all $e = (R, \Omega), \overline{e} = (\overline{R}, \overline{\Omega}) \in \mathscr{E}^N$, if $R_M = \overline{R}_M$ and $\sum_M \varphi_i(e) = \sum_M \varphi_i(\overline{e})$, then for all $i \in M$, $\varphi_i(e) = \varphi_i(\overline{e})$.

Separability was introduced by Moulin (1987) in the context of surplus sharing. Chun (1998a,b,1999) studies *separability* in the contexts of economies with single-peaked preferences, quasi-linear choice, and bankruptcy.

It follows easily that *consistency* implies *separability*.

Lemma 3 *Let φ be a rule that satisfies consistency. Then φ satisfies separability.*

Proof. The proof is essentially “model-free”. See for instance Chun (1998a), proof of Lemma 6. □

Next, we show that *separability* implies *non-bossiness*. A rule is called bossy if an agent, by changing his announcement, can affect the allotments of the remaining agents without changing his own allotment. A rule that does not allow for this kind of influence is called *non-bossy*. This concept of *(non-)bossiness* was introduced by Satterthwaite and Sonnenschein (1981).

Non-bossiness: For all $N \in \mathscr{P}$, all $e = (R, \Omega), \overline{e} = (\overline{R}, \Omega) \in \mathscr{E}^N$, and all $j \in N$, if $\overline{R}$ is a j-deviation from R and $\varphi_j(e) = \varphi_j(\overline{e})$, then for all $i \in N \backslash \{j\}$, $\varphi_i(e) = \varphi_i(\overline{e})$.

In Klaus (1999) we consider the following weakening of *non-bossiness*. *Weak non-bossiness in terms of welfare* states that no agent can influence some other agent’s welfare without affecting his own allotment.

Weak non-bossiness in terms of welfare: For all $N \in \mathscr{P}$, all $e = (R, \Omega), \bar{e} = (\bar{R}, \Omega) \in \mathscr{E}^N$, and all $j \in N$, if $\bar{R}$ is a j-deviation from R and $\varphi_j(e) = \varphi_j(\bar{e})$, then for all $i \in N \setminus \{j\}$, $\varphi_i(e)\, I_i\, \varphi_i(\bar{e})$.

In the sequel we use the shorter phrase of *weak non-bossiness* for *weak non-bossiness in terms of welfare*. For a further discussion of *weak non-bossiness* for this model we refer to Klaus (1999). It is obvious that *non-bossiness* implies *weak non-bossiness*. We prove next, that *separability* implies *non-bossiness*.

Lemma 4 *Let φ be a rule that satisfies separability. Then φ satisfies non-bossiness.*

Note that the proof of the lemma is essentially “model-free”.

Proof. Let φ satisfy *separability* and let $N \in \mathscr{P}$, $e = (R, \Omega), \bar{e} = (\bar{R}, \Omega) \in \mathscr{E}^N$, and $j \in N$. Furthermore, let $\bar{R}$ be a j-deviation from R and $\varphi_j(e) = \varphi_j(\bar{e})$. Hence, $R_{N\setminus\{j\}} = \bar{R}_{N\setminus\{j\}}$ and $\sum_{N\setminus\{j\}} \varphi_i(e) = \sum_{N\setminus\{j\}} \varphi_i(\bar{e})$. Thus, by *separability*, for all $i \in N\setminus\{j\}$, $\varphi_i(e) = \varphi_i(\bar{e})$. This proves *non-bossiness*. $\square$

Before we characterize the class of rules that satisfy *Pareto efficiency*, *strategy-proofness*, and either *separability* or *non-bossiness*, we recall the characterization of the class of *Pareto efficient*, *strategy-proof* rules that satisfy either *weak non-bossiness* or *coalitional strategy-proofness*; see Klaus (1999).

Let $N \in \mathscr{P}$ and $e \in \mathscr{E}^N$. A rule that satisfies *Pareto efficiency*, *strategy-proofness*, and *weak non-bossiness*, assigns the whole endowment to a single agent. The selection of this agent can be described as follows.

We fix a linear order of the set of agents and ask the agent who is first in this order whether he prefers Ω to 0. If he does, he receives it and we are done. If not, we ask the second agent the same question; *etc.*. Hence, the first agent according to the fixed order who prefers the whole endowment to receiving nothing receives it.

If no agent strictly prefers Ω to 0 and at least one agent is indifferent between 0 and Ω, then the endowment can be assigned to any of the indifferent agents.

If all agents strictly prefer 0 to Ω, there exists a preselected agent who receives it. Loosely speaking, he is the “scapegoat”.

For a formal description, we need some extra notation. Let $N \in \mathscr{P}$ and $e \in \mathscr{E}^N$. A *permutation π^N on N* is a bijective function $\pi^N : N \to N$. By Π^N we denote the *set of all permutations on N*. Since for $\pi^N \in \Pi^N$ and $i \in N$, $\pi^N(i)$ can also be interpreted as the position of i in a linear order on N, we alternatively refer to π^N as to the *linear order on N*. By G^N we denote the set of *choice functions* $g^N : \mathscr{E}^N \to N$ such that $g^N(e) \in N_{0,\Omega}(e)$ if $N_{0,\Omega}(e) \neq \emptyset$.

Selection $\mathbf{f_{\pi^N, g^N, k^N}}$: Let $N \in \mathscr{P}$ and let $\pi^N \in \Pi^N$, $g^N \in G^N$, and $k^N \in N$. Then the *selection* $f_{\pi^N, g^N, k^N} : \mathscr{E}^N \to N$ (based on π^N, g^N, and k^N) is defined as follows. Let $e \in \mathscr{E}^N$.

Case 1: If $N_\Omega(e) \neq \emptyset$, then $f_{\pi^N,g^N,k^N}(e) = \arg\min\{\pi^N(i) \mid i \in N_\Omega(e)\}$.

Case 2: If $N_\Omega(e) = \emptyset$ and $N_{0,\Omega}(e) \neq \emptyset$, then $f_{\pi^N,g^N,k^N}(e) = g^N(e)$.

Case 3: If $N = N_0(e)$, then $f_{\pi^N,g^N,k^N}(e) = k^N$.

Note that in Case 1 the selection rule equals a serial-dictatorship. Also Case 3 can be interpreted as (degenerated) serially dictatorial.

Theorem 1 *Let φ be a rule. Then the following statements are equivalent.*
(i) *φ satisfies Pareto efficiency and coalitional strategy-proofness.*
(ii) *φ satisfies Pareto efficiency, strategy-proofness, and weak non-bossiness.*
(iii) *For all $N \in \mathscr{P}$ and all $\Omega \in \mathbb{R}_+$ there exist $\pi^N \in \Pi^N$, $g^N \in G^N$, and $k^N \in N$ such that for all $e = (R, \Omega) \in \mathscr{E}^N$, φ assigns Ω to agent $f_{\pi^N,g^N,k^N}(e)$, i.e.,*

$$\varphi_{f_{\pi^N,g^N,k^N}(e)}(e) = \Omega. \tag{1}$$

Proof. See Klaus (1999), proof of Theorem 1. □

Since *separability* implies *non-bossiness* (Lemma 4), and therefore *weak non-bossiness*, the class of rules that satisfy *Pareto efficiency*, *strategy-proofness*, and either *separability* or *non-bossiness* must be contained in the class of rules as described in Theorem 1; in fact, it is a strict subclass. The difference lies in the selection of the agent who receives the whole endowment when none of the agents strictly prefers the endowment above receiving nothing and there are agents that are indifferent. In this case, the selection of the agent must be *non-bossy*, *i.e.*, if an agent that is not selected changes his preference relation in such a way that he is still not selected, then this deviation must not change the original selection (a change of the selection after the unilateral deviation of the agent who is not receiving the endowment would be bossy). In order to satisfy *separability* and *non-bossiness* for rules as described in Theorem 1 we restrict the class of choice functions that are admissable for the selection as follows. By $\overline{G}^N$ we denote the set of *non-bossy choice functions* $\overline{g}^N : \mathscr{E}^N \to N$, *i.e.*, $\overline{g}^N \in G^N$ and for all $e = (R, \Omega), \overline{e} = (\overline{R}, \Omega) \in \mathscr{E}^N$, and all $j \in N$, if $\overline{R}$ is a j-deviation from R, $j \neq \overline{g}^N(e)$, and $j \neq \overline{g}^N(\overline{e})$, then $\overline{g}^N(e) = \overline{g}^N(\overline{e})$.

Theorem 2 *Let φ be a rule. Then the following statements are equivalent.*
(i) *φ satisfies Pareto efficiency, strategy-proofness, and separability.*
(ii) *φ satisfies Pareto efficiency, strategy-proofness, and non-bossiness.*
(iii) *For all $N \in \mathscr{P}$ and all $\Omega \in \mathbb{R}_+$ there exist $\pi^N \in \Pi^N$, $g^N \in \overline{G}^N$, and $k^N \in N$ such that for all $e = (R, \Omega) \in \mathscr{E}^N$, φ assigns Ω to agent $f_{\pi^N,g^N,k^N}(e)$, i.e.,*

$$\varphi_{f_{\pi^N,g^N,k^N}(e)}(e) = \Omega. \tag{2}$$

Proof. Assume that φ satisfies *Pareto efficiency*, *strategy-proofness*, and *separability*. By Lemma 4 it follows that φ satisfies *non-bossiness*. Hence, **(i)** implies **(ii)**.

Assume that φ satisfies *Pareto efficiency, strategy-proofness,* and *non-bossiness*. Then φ satisfies *weak non-bossiness* and by Theorem 1, for all $N \in \mathscr{P}$ and all $\Omega \in \mathbb{R}_+$ there exist $\pi^N \in \Pi^N$, $g^N \in G^N$, and $k^N \in N$ such that for all $e = (R, \Omega) \in \mathscr{E}^N$, $\varphi_{f_{\pi^N, g^N, k^N}(e)}(e) = \Omega$. *Non-bossiness* implies that g^N must be a *non-bossy* selection. Thus, $g^N \in \overline{G}^N$. Hence, **(ii)** implies **(iii)**.

Let $N \in \mathscr{P}$, $\Omega \in \mathbb{R}_+$, $\pi^N \in \Pi^N$, $g^N \in \overline{G}^N$, and $k^N \in N$ be such that for all $e = (R, \Omega) \in \mathscr{E}^N$, $\varphi_{f_{\pi^N, g^N, k^N}(e)}(e) = \Omega$. It is easy, but tedious, to prove that φ satisfies *Pareto efficiency, strategy-proofness,* and *separability*. We leave this part of the proof to the reader. Hence, **(iii)** implies **(i)**. □

4 Population-monotonicity

In this section we study the solidarity property *population-monotonicity*. It incorporates a notion of solidarity among agents when changes in the population occur, e.g., if a group of agents leave, then, after this change, either all remaining agents are made (weakly) better off or they all are made (weakly) worse off.

Population-monotonicity: For all $N, M \in \mathscr{P}$, all $e = (R, \Omega) \in \mathscr{E}^N$, and all $\overline{e} = (\overline{R}, \Omega) \in \mathscr{E}^M$, if $M \subsetneq N$ and $R_M = \overline{R}$, then either [for all $i \in M$, $\varphi_i(e)\, R_i\, \varphi_i(\overline{e})$] or [for all $i \in M$, $\varphi_i(\overline{e})\, R_i\, \varphi_i(e)$].

Thomson (1983) introduced *population-monotonicity* in the context of bargaining. The notion of *population-monotonicity* we introduce above is due to Chun (1986) who calls it "solidarity". For a survey on *population-monotonicity* we refer to Thomson (1995b).

First, we show that a rule φ that satisfies *Pareto efficiency, strategy-proofness,* and *population-monotonicity* assigns the whole endowment to a single agent.

Let $\mathscr{B}_\varphi = \{e = (R, \Omega) \in \mathscr{E}^N \mid N \in \mathscr{P} \text{ and for all } i \in N, \varphi_i(e) \neq \Omega\}$ be the set of economies that yield a *broken allocation* under φ, *i.e.*, a feasible allocation where no agent obtains the whole endowment.

Lemma 5 *Let φ be a rule that satisfies Pareto efficiency, strategy-proofness, and population-monotonicity. Then, $\mathscr{B}_\varphi = \emptyset$.*

Proof. Let φ satisfy the properties listed in the lemma and suppose by contradiction that $\mathscr{B}_\varphi \neq \emptyset$.

Since $\mathscr{B}_\varphi \neq \emptyset$, there exist $N \in \mathscr{P}$, $e = (R, \Omega) \in \mathscr{E}^N$, and $i, j \in N$ such that $\varphi_i(e) \neq 0$ and $\varphi_j(e) \neq 0$. By the Efficiency Lemma, either **(a)** $\varphi_i(e)\, P_i\, 0$ and $\varphi_j(e)\, P_j\, 0$ or **(b)** $\varphi_i(e)\, P_i\, \Omega$ and $\varphi_j(e)\, P_j\, \Omega$.

Now, let $M = \{i, j\}$ and consider $\overline{e} = (R_M, \Omega) \in \mathscr{E}^M$. By *population-monotonicity,* either [$\varphi_i(e)\, R_i\, \varphi_i(\overline{e})$ and $\varphi_j(e)\, R_j\, \varphi_j(\overline{e})$] or [$\varphi_i(\overline{e})\, R_i\, \varphi_i(e)$ and $\varphi_j(\overline{e})\, R_j\, \varphi_j(e)$].

Since $\overline{e}$ is a two-agent economy, by Lemma 2, it follows that either [$\varphi_i(\overline{e}) = \Omega$ and $\varphi_j(\overline{e}) = 0$] or [$\varphi_i(\overline{e}) = 0$ and $\varphi_j(\overline{e}) = \Omega$]. This is in contradiction to **(a)** and **(b).** □

Lemma 6 *Let φ be a rule that satisfies Pareto efficiency, strategy-proofness, and population-monotonicity. Then φ satisfies weak non-bossiness.*

Proof. Let φ satisfy *Pareto efficiency, strategy-proofness,* and *population-monotonicity*. Let $e = (R, \Omega), \overline{e} = (\overline{R}, \Omega) \in \mathscr{E}^N$, and $j \in N$ be such that $\overline{R}$ is a j-deviation from R and $\varphi_j(e) = \varphi_j(\overline{e})$. By Lemma 5, $\mathscr{B}_\varphi = \emptyset$. Hence, either $\varphi_j(e) = \varphi_j(\overline{e}) = 1$ or $\varphi_j(e) = \varphi_j(\overline{e}) = 0$. In order to prove *weak non-bossiness,* we have to show that for all $i \in N\backslash\{j\}$, $\varphi_i(e)\, I_i\, \varphi_i(\overline{e})$. If $\varphi_j(e) = \varphi_j(\overline{e}) = 1$, then for all $i \in N\backslash\{j\}$, $\varphi_i(e) = \varphi_i(\overline{e}) = 0$. Let $\varphi_j(e) = \varphi_j(\overline{e}) = 0$. Suppose, by contradiction, that there exist $k, l \in N\backslash\{j\}$ such that $\varphi_k(e)\, P_k\, \varphi_k(\overline{e})$ and $\varphi_l(\overline{e})\, P_l\, \varphi_l(e)$. Hence, by *Pareto efficiency,* either $k, l \in N_\Omega(e)$ or $k, l \in N_0(e)$. First, consider $e_{N\backslash\{j\}} = (R_{N\backslash\{j\}}, \Omega) \in \mathscr{E}^N$. By *population-monotonicity,* $\varphi_k(e_{N\backslash\{j\}}) = \varphi_k(e)$ and $\varphi_l(e_{N\backslash\{j\}}) = \varphi_l(e)$. Next, consider $\overline{e}_{N\backslash\{j\}} = (\overline{R}_{N\backslash\{j\}}, \Omega) \in \mathscr{E}^N$. By *population-monotonicity,* $\varphi_k(\overline{e}_{N\backslash\{j\}}) = \varphi_k(\overline{e})$ and $\varphi_l(\overline{e}_{N\backslash\{j\}}) = \varphi_l(\overline{e})$. Thus, $\varphi_k(e_{N\backslash\{j\}})\, P_k\, \varphi_k(\overline{e}_{N\backslash\{j\}})$ and $\varphi_l(\overline{e}_{N\backslash\{j\}})\, P_l\, \varphi_l(e_{N\backslash\{j\}})$. This is in contradiction to $e_{N\backslash\{j\}} = \overline{e}_{N\backslash\{j\}}$. □

Next, we characterize the class of *Pareto efficient, strategy-proof,* and *population-monotonic* rules.

As proven in Lemma 5, a rule that satisfies *Pareto efficiency, strategy-proofness,* and *population-monotonicity,* assigns the whole endowment to a single agent. The selection of this agent can be described as follows. Let $N \in \mathscr{P}$ be a set of agents and $e = (R, \Omega) \in \mathscr{E}^N$ an economy.

By *Pareto efficiency,* the endowment Ω should be allotted to an agent who prefers Ω to 0, if there is such an agent. Among these agents, if there are several, the choice is made with respect a linear order π_+ on $\mathbb{P}$, the set of potential agents.

We ask the first agent according to π_+ who is a member of N, whether he prefers Ω to 0. If he does, he receives it and we are done. If not, we ask the second agent according to π_+ who is a member of N the same question; *etc.*. Hence, the first agent in the order π_+ who is in N and prefers the endowment to receiving nothing receives it.

If no agent in N strictly prefers Ω to 0 and at least one agent in N is indifferent between 0 and Ω, then the endowment is assigned to an agent who is indifferent between 0 and Ω.

Finally, if all agents in N strictly prefer 0 to Ω, then Ω is assigned to an agent with respect to a linear order π_- on $\mathbb{P}$. Now, the first agent in the order who is in N receives the endowment.

For a formal description, we introduce the following notation.

Let $\Pi^{\mathbb{P}}$ denote the set of all permutations, or *linear orders*, on $\mathbb{P}$. We call $\mathscr{G} := \{g^N \in G^N\}_{N\in\mathscr{P}}$ a *collection of choice functions*.

Selection $\mathbf{f}_{\pi_+,\mathscr{G},\pi_-}$: Let $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$ and $\mathscr{G} = \{g^N \in G^N\}_{N\in\mathscr{P}}$. Then, the *selection $f_{\pi_+,\mathscr{G},\pi_-}$* (based on π_+, π_-, and $\mathscr{G}$) is defined as follows. Let $N \in \mathscr{P}$ and $e = (R, \Omega) \in \mathscr{E}^N$.

Case 1: If $N_\Omega(e) \neq \emptyset$, then $f_{\pi_+,\mathscr{G},\pi_-}(e) = \arg\min\{\pi_+(i) \mid i \in N_\Omega(e)\}$.

Case 2: If $N_\Omega(e) = \emptyset$ and $N_{0,\Omega}(e) \neq \emptyset$, then $f_{\pi_+,\mathscr{G},\pi_-}(e) = g^N(e)$.

Case 3: If $N_0(e) = N$, then $f_{\pi_+,\mathscr{G},\pi_-}(e) = \arg\min\{\pi_-(i) \mid i \in N_0(e)\}$.

Theorem 3 *A rule φ satisfies Pareto efficiency, strategy-proofness, and population-monotonicity if and only if for all $\Omega \in \mathbb{R}_+$ there exist $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$ and $\mathscr{G} = \{g^N \in G^N\}_{N\in\mathscr{P}}$ such that for all $N \in \mathscr{P}$ and all $e = (R, \Omega) \in \mathscr{E}^N$, φ assigns Ω to agent $f_{\pi_+,\mathscr{G},\pi_-}(e)$, i.e.,*

$$\varphi_{f_{\pi_+,\mathscr{G},\pi_-}(e)}(e) = \Omega. \tag{3}$$

Proof. Assume that φ satisfies *Pareto efficiency*, *strategy-proofness*, and *population-monotonicity*. By Lemma 6 it follows that φ satisfies *weak non-bossiness*. Hence, by Theorem 1, for all $N \in \mathscr{P}$ and all $\Omega \in \mathbb{R}_+$ there exist $\pi^N \in \Pi^N$, $g^N \in G^N$, and $k^N \in N$ such that for all $e = (R, \Omega) \in \mathscr{E}^N$,

$$\varphi_{f_{\pi^N,g^N,k^N}(e)}(e) = \Omega. \tag{4}$$

Let $\Omega \in \mathbb{R}_+$. We prove that there exist $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$ and $\mathscr{G} = \{g^N \in G^N\}_{N\in\mathscr{P}}$ such that for all $N \in \mathscr{P}$ and all $e = (R, \Omega) \in \mathscr{E}^N$, $f_{\pi_+,\mathscr{G},\pi_-}(e) = f_{\pi^N,g^N,k^N}(e)$ and $\varphi_{f_{\pi_+,\mathscr{G},\pi_-}(e)}(e) = \Omega$.

(a) Definition of $\pi_+ \in \Pi^{\mathbb{P}}$.

We prove that $\pi_+ = \pi^{\mathbb{P}} \in \Pi^{\mathbb{P}}$ where $\pi^{\mathbb{P}}$ is the order obtained from (4) for $N = \mathbb{P}$. Suppose, by contradiction, that there exist $N \in \mathscr{P}$ and $e = (R, \Omega) \in \mathscr{E}^N$ such that $N_\Omega(e) \neq \emptyset$ and $f_{\pi^N,g^N,k^N}(e) = \arg\min\{\pi^N(i) \mid i \in N_\Omega(e)\} \neq \arg\min\{\pi^{\mathbb{P}}(i) \mid i \in N_\Omega(e)\}$). Hence, $N \subsetneq \mathbb{P}$. Consider the economy $\overline{e} = (\overline{R}, \Omega) \in \mathscr{E}^{\mathbb{P}}$ such that $\overline{R}_N = R$ and $N_\Omega(\overline{e}) = N_\Omega(e)$. Then, by (4), $\varphi_{f_{\pi^{\mathbb{P}},g^{\mathbb{P}},k^{\mathbb{P}}}(\overline{e})}(\overline{e}) = \Omega$ and $f_{\pi^{\mathbb{P}},g^{\mathbb{P}},k^{\mathbb{P}}}(\overline{e}) = \arg\min\{\pi^{\mathbb{P}}(i) \mid i \in N_\Omega(\overline{e})\}) = \arg\min\{\pi^{\mathbb{P}}(i) \mid i \in N_\Omega(e)\}$.

By *population-monotonicity*, $f_{\pi^{\mathbb{P}},g^{\mathbb{P}},k^{\mathbb{P}}}(\overline{e}) = f_{\pi^N,g^N,k^N}(e)$. Hence, $\arg\min\{\pi^{\mathbb{P}}(i) \mid i \in N_\Omega(\overline{e})\} = \arg\min\{\pi^N(i) \mid i \in N_\Omega(e)\}$. This is a contradiction.

(b) Definition of $\mathscr{G}$.

For each $N \in \mathscr{P}$ consider $g^N \in G^N$ as described in (4). Then, the collection of choice functions $\mathscr{G}$ equals $\{g^N \in G^N\}_{N\in\mathscr{P}}$.

(c) Definition of $\pi_- \in \Pi^{\mathbb{P}}$.

By (4), for all $e = (R, \Omega) \in \mathscr{E}^{\mathbb{P}}$ such that $N_0(e) = \mathbb{P}$, $\varphi_{k^{\mathbb{P}}}(e) = \Omega$. Set $\pi_-(k^{\mathbb{P}}) = 1$. Next, consider $N \subsetneq \mathbb{P}$ such that $k^{\mathbb{P}} \in N$ and $\overline{e} = (\overline{R}, \Omega) \in \mathscr{E}^N$ such that $N_0(\overline{e}) = N$. Then, by *population-monotonicity*, $\varphi_{k^{\mathbb{P}}}(\overline{e}) = \Omega$.

Next, let $P^1 := \mathbb{P}\backslash\{k^{\mathbb{P}}\}$. Then, by (4), for all $e = (R, \Omega) \in \mathscr{E}^{P^1}$ such that $N_0(e) = P^1$, $\varphi_{k^{P^1}}(e) = \Omega$. Set $\pi_-(k^{P^1}) = 2$. Similarly as before it follows that for all $N \subsetneq P^1$ such that $k^{P^1} \in N$ and $\overline{e} = (\overline{R}, \Omega) \in \mathscr{E}^N$ such that $N_0(\overline{e}) = N$, $\varphi_{k^{P^1}}(\overline{e}) = \Omega$.

Let $P^2 := P^1\backslash\{k^{P^1}\}$. Then, by (4), for all $e = (R, \Omega) \in \mathscr{E}^{P^2}$ such that $N_0(e) = P^2$, $\varphi_{k^{P^2}}(e) = \Omega$. Set $\pi_-(i_3) = 3$.

It is now clear how the linear order $\pi_- \in \Pi^{\mathbb{P}}$ is constructed step by step by leaving out the agent receiving the endowment. By the definition of π_-, it follows for all $N \in \mathscr{P}$ and all $e = (R, \Omega) \in \mathscr{E}^N$ such that $N_0(e) = N$, $\varphi_{f_{\pi^N, g^N, k^N}(e)}(e) = \Omega$ and $f_{\pi^N, g^N, k^N}(e) = \arg\min\{\pi_-(i) \mid i \in N_0(R)\}$.

(d) By (a), (b), and (c) it follows easily, that for each $N \in \mathscr{P}$ and all $e = (R, \Omega) \in \mathscr{E}^N$, $f_{\pi_+, \mathscr{G}, \pi_-}(e) = f_{\pi^N, g^N, k^N}(e)$ and $\varphi_{f_{\pi_+, \mathscr{G}, \pi_-}(e)}(e) = \Omega$.

Let $\Omega \in \mathbb{R}_+$. Let $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$ and $\mathscr{G} = \{g^N \in G^N\}_{N \in \mathscr{P}}$ be such that for all $N \in \mathscr{P}$ and all $e = (R, \Omega) \in \mathscr{E}^N$, $\varphi_{f_{\pi_+, \mathscr{G}, \pi_-}(e)}(e) = \Omega$. It is easy, but tedious, to prove that φ satisfies *Pareto efficiency*, *strategy-proofness*, and *population-monotonicity*. We leave this part of the proof to the reader. □

Remark 1 Note that any *Pareto efficient*, *strategy-proof*, and *population-monotonic* rule as described in Theorem 3 in addition satisfies *coalitional strategy-proofness* and *weak non-bossiness*. However, a rule as described in Theorem 3 does not necessarily satisfy *separability* or *non-bossiness* (see Theorem 2). It is also clear that a rule that satisfies *Pareto efficiency*, *strategy-proofness*, and either *non-bossiness* or *separability* (see Theorem 2) does not necessarily satisfy *population-monotonicity*.

5 Other properties and logical relations

First, we show that the classes of rules that we characterize in Sections 3 and 4 are non-empty. In the following example, we describe a rule that satisfies all properties stated in the theorems.

Example 1 The following rule φ satisfies *Pareto efficiency, (coalitional) strategy-proofness, (weak) non-bossiness, separability,* and *population-monotonicity*. For all $N \in \mathscr{P}$ and all $e = (R, \Omega) \in \mathscr{E}^N$:

Case 1 : *If $N_\Omega(e) \neq \emptyset$, then $\varphi^{\min}$ assigns Ω to agent* $\min\{i \mid i \in N_\Omega(e)\}$.

Case 2 : *If $N_\Omega(e) = \emptyset$, $N_{0,\Omega}(e) \neq \emptyset$, then $\varphi^{\min}$ assigns Ω to agent* $\min\{i \mid i \in N_{0,\Omega}(e)\}$.

Case 3 : *If $N = N_0(e)$, then $\varphi^{\min}$ assigns Ω to agent* $\min\{i \mid i \in N_0(e)\}$.

By id$\in \Pi^{\mathbb{P}}$ we denote the identity permutation defined by id(i) = i *for all $i \in \mathbb{P}$. Let $\pi_+ = \pi_- = \mathrm{id} \in \Pi^{\mathbb{P}}$ and define $g^N_{\min} \in G^N$ such that for all $N \in \mathscr{P}$ and all $e = (R, \Omega) \in \mathscr{E}^N$, $g^N_{\min}(e) = \min\{i \mid i \in N_{0,\Omega}(e)\}$ if $N_{0,\Omega}(e) \neq \emptyset$ and $g^N_{\min}(e) = \min\{i \mid i \in N\}$ otherwise. Let $\mathscr{G}' = \{g^N_{\min}\}_{N \in \mathscr{P}}$. Now choosing for all $\Omega \in \mathbb{R}_+$ the selection $f_{\mathrm{id}, \mathscr{G}', \mathrm{id}}$ yields for all $N \in \mathscr{P}$ and all $e = (R, \Omega) \in \mathscr{E}^N$, $\varphi^{\min}(e) = \varphi_{f_{\mathrm{id}, \mathscr{G}', \mathrm{id}}(e)}(e) = \Omega$.* ♢

The following examples show that the characterizations given in Theorems 2 and 3 are tight, *i.e.*, dropping any of the properties yields alternative rules.

Example 2 The following rule φ satisfies *(coalitional) strategy-proofness, (weak) non-bossiness, separability,* and *population-monotonicity*, but not *Pareto efficiency*. For all $N \in \mathscr{P}$ and all $e = (R, \Omega) \in \mathscr{E}^N$, φ assigns Ω to agent $\min\{i \mid i \in N\}$. ♢

Example 3 The following rule φ satisfies *Pareto efficiency, (weak) non-bossiness, separability,* and *population-monotonicity,* but not *(coalitional) strategy-proofness*. For all $N \in \mathscr{P}$ and all $e = (R, \Omega) \in \mathscr{E}^N$ such that $N_\Omega(e) \neq \emptyset$, let $N^1_\Omega(e) = \{i \in N_\Omega(e) \mid d(R_i) = 0\}$ and $N^2_\Omega(e) = \{i \in N_\Omega(e) \mid d(R_i) \neq 0\}$. Then, for all $N \in \mathscr{P}$ *and* all $e = (R, \Omega) \in \mathscr{E}^N$, we define $\varphi(e)$ as follows. If $N^1_\Omega(e) \neq \emptyset$, then φ assigns Ω to agent $\min\left\{i \mid i \in N^1_\Omega(e)\right\}$. If $N^1_\Omega(e) = \emptyset$ and $N^2_\Omega(e) \neq \emptyset$, then φ assigns Ω to agent $\min\left\{i \mid i \in N^2_\Omega(e)\right\}$. For all remaining $e = (R, \Omega) \in \mathscr{E}^N$, $\varphi(e) = \varphi^{\min}(e)$. ♢

Example 4 The following rule $\tilde{\varphi}$ satisfies *Pareto efficiency, strategy-proofness,* but not *(weak) non-bossiness,* not *separability,* and not *population-monotonicity.* For $T = \{1, 2, 3\}$ *and* $e = (R, 1) \in \mathscr{E}^T$, *define* $\tilde{\varphi}(e)$ *as follows. If* $1\, P_3\, 0$, *then*

$$\tilde{\varphi}(e) = \begin{cases} (\frac{1}{2}, \frac{1}{2}, 0) & \textit{if } 1\, P_1\, \frac{1}{2}\, P_1\, 0 \textit{ and } 1\, P_2\, \frac{1}{2}\, P_2\, 0, \\ (0, 0, 1) & \textit{otherwise.} \end{cases} \tag{5}$$

For all remaining $e = (R, \Omega) \in \mathscr{E}^N$, $\tilde{\varphi}(e) = \varphi^{\min}(e)$. ♢

As shown in Klaus, Peters, and Storcken (1997) and in Klaus (1998) the class of rules described in Theorem 3 almost equals the class of rules that satisfy *Pareto efficiency, strategy-proofness,* and *consistency*. The only difference in the description of the class of *Pareto efficient, strategy-proof,* and *consistent* rules is a restriction of the admissible collections of choice rules. If a rule satisfies *consistency,* then also its collection of choice rules must be "consistent".

Next, we discuss a property that is related to *consistency,* namely *converse consistency*. *Converse consistency* determines the desirability of an allocation on the basis of the desirability of its restrictions to all two-agent reduced economies.

Converse consistency: For all $N \in \mathscr{P}$, all $e = (R, \Omega) \in \mathscr{E}^N$, and all feasible allocations $x \in \mathbb{R}^N_+$, if for all $M \subseteq N$ with $|M| = 2$, $\varphi(R_M, \sum_M x_i) = x_M$, then $\varphi(e) = x$.

The property of *converse consistency* is reviewed in Thomson (1996).

As the following example shows, a *Pareto efficient, strategy-proof,* and *conversely consistent* rule does not necessarily always assign the whole endowment to a single agent.

Example 5 The following allocation rule φ satisfies *Pareto efficiency, strategy-proofness,* and *converse consistency*. Let $T_1 = \{1, 2\}$, $T_2 = \{2, 3\}$, and $T_3 = \{1, 3\}$. For all $e_1 = (R^1, \Omega) \in \mathscr{E}^{T_1}$ such that $N_\Omega(e_1) = T_1$, let $\varphi_1(e_1) = \Omega$, for all $e_2 = (R^2, \Omega) \in \mathscr{E}^{T_2}$ such that $N_\Omega(e_2) = T_2$, let $\varphi_2(e_2) = \Omega$, and for all $e_3 = (R^3, \Omega) \in \mathscr{E}^{T_3}$ such that $N_\Omega(e_3) = T_3$, let $\varphi_3(e_3) = \Omega$. For all remaining economies $e = (R, \Omega) \in \mathscr{E}^N$, $N \in \mathscr{P}$, let $\varphi(e) = \tilde{\varphi}(e)$, where $\tilde{\varphi}$ is defined in Example 4.

The proof that φ satisfies *converse consistency* is easy. It is based on the following two cases.

Case 1: The allocation assigned for an economy $e \in \mathscr{E}^N$, where $N \in \mathscr{P}$ and $|N| > 2$, is consistent with the choice for all two-agent economies (if $\{1,2,3\} \cap N \neq \{1,2,3\}$).

Case 2: There exists no allocation that satisfies the hypothesis of converse consistency, namely, that the allocation for each reduced two-agent economy equals the restriction of the allocation to this set (this follows from the "cycle" we constructed for the two-agent subsets of $\{1,2,3\}$). ♢

We conclude this section with the discussion of the solidarity property *resource-monotonicity*. *Resource-monotonicity* describes the effect of a change in the endowment on the welfare of the agents. If after such a change either all agents (weakly) lose together or all (weakly) gain together, then the rule satisfies *resource-monotonicity*.

Resource-monotonicity: For all $N \in \mathscr{N}$, all $e = (R, \Omega) \in \mathscr{E}^N$, and all $\overline{e} = (R, \overline{\Omega}) \in \mathscr{E}^N$, either [for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(\overline{e})$] or [for all $i \in N$, $\varphi_i(\overline{e})\, R_i\, \varphi_i(e)$].

Conditions of *resource-monotonicity* have been studied by Chun and Thomson (1988), Moulin and Thomson (1988), Roemer (1986), and Thomson (1978, 1994).

The following example shows that *Pareto efficiency*, *strategy-proofness*, and *resource-monotonicity* are not compatible.

Example 6 Suppose the rule φ satisfies *Pareto efficiency, strategy-proofness*, and *resource-monotonicity*. Let $N = \{1,2\}$, $\Omega = 1$, $\overline{\Omega} = 2$, and $e = (R, \Omega) \in \mathscr{E}^N$ be such that $2\, P_1\, 1\, P_1\, 0$ and $2\, P_2\, 0\, P_2\, 1$. Consider $e = (R, \Omega)$. By *Pareto efficiency*, $\varphi_1(e) = \Omega$. Next, consider $e' = (R, \overline{\Omega})$. By Lemma 2, $\varphi_i(e') = \overline{\Omega}$ for some $i \in N$. Then, by *resource-monotonicity*, $\varphi_1(e') = \overline{\Omega}$. Now, consider the 2-deviation $\overline{R}$ from R such that $\overline{R}_2 = R_1$. Consider $\overline{e} = (\overline{R}, \overline{\Omega})$. Then, by *strategy-proofness*, $\varphi_1(\overline{e}) = \overline{\Omega}$.

Next, consider $\tilde{R} \in \mathscr{D}^N$ such that $\tilde{R}_1 = R_2$ and $\tilde{R}_2 = R_1$. Consider $\tilde{e} = (\tilde{R}, \Omega)$. By *Pareto efficiency*, $\varphi_2(\tilde{e}) = \Omega$. Next, consider $\tilde{e}' = (\tilde{R}, \overline{\Omega})$. By Lemma 2, $\varphi_i(\tilde{e}') = \overline{\Omega}$ for some $i \in N$. Then, by *resource-monotonicity*, $\varphi_2(\tilde{e}') = \overline{\Omega}$. Now, $\overline{R}$ is a 1-deviation from $\tilde{R}$ such that $\overline{R}_1 = \tilde{R}_2 = R_1$. Consider $\overline{e} = (\overline{R}, \overline{\Omega})$. Then, by *strategy-proofness*, $\varphi_2(\overline{e}) = \overline{\Omega}$. This is in contradiction to $\varphi_1(\overline{e}) = \overline{\Omega}$. ♢

6 The assignment of an indivisible object

Consider the well-known problem of allocating an indivisible commodity or object among a group of agents, e.g., a task or a real object. Obviously, this problem is closely related to the allocation problem with single-dipped preferences we introduced in Section 2 (see Lemmas 2 and 5 and Theorems 1, 2, and 3).

In order to keep this section self-contained, we briefly introduce the model. An indivisible object Ω has to be allocated among a non-empty and finite set

$N \in \mathscr{P}$ of agents. Each agent $i \in N$ is equipped with a preference relation R_i defined over the two alternatives "receiving nothing", denoted by 0, and "receiving the object", denoted by Ω. Hence, for each agent $i \in N$ either $0\, P_i\, \Omega$, $0\, I_i\, \Omega$, or $\Omega\, P_i\, 0$. By $\mathscr{R}_{\{0,\Omega\}}$ we denote the set of preference relations over $\{0, \Omega\}$ and $\mathscr{R}^N_{\{0,\Omega\}}$ denotes the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathscr{R}_{\{0,\Omega\}}$. Thus, the class of all economies is denoted by $\mathscr{E} = \cup_{N \in \mathscr{P}} \mathscr{R}^N_{\{0,\Omega\}}$.

Let $N \in \mathscr{P}$. A *feasible allocation for* $R \in \mathscr{R}^N_{\{0,\Omega\}}$ is an assignment of the object Ω to exactly one of the agents $i \in N$. Note that it is without loss of generality that free disposal of the commodity is not allowed. An *assignment rule* φ is a function that assigns to every $R \in \mathscr{R}^N_{\{0,\Omega\}}$ a feasible allocation, denoted $\varphi(R)$. Note that either $\varphi_i(R) = 0$ or $\varphi_i(R) = \Omega$. Properties of assignment rules and further notation are as defined in the previous sections.

It is easy to show that Theorems 2 and 3 remain true for assignment rules. Before stating this result as a corollary, we note that for the simple model we consider here, *Pareto efficiency* implies *strategy-proofness*.

Lemma 7 *Let φ be an assignment rule that satisfies Pareto efficiency. Then φ satisfies strategy-proofness.*

Proof. See Klaus (1999), proof of Lemma 5. □

Corollary 1 *Let φ be an assignment rule. Then the following statements are equivalent.*

(i) *φ satisfies Pareto efficiency and separability.*

(ii) *φ satisfies Pareto efficiency and non-bossiness.*

(iii) *For all $N \in \mathscr{P}$ there exist $\pi^N \in \Pi^N$, $g^N \in \overline{G}^N$, and $k^N \in N$ such that for all $R \in \mathscr{R}^N_{\{0,\Omega\}}$, φ assigns Ω to agent $f_{\pi^N,g^N,k^N}(R)$, i.e.,*

$$\varphi_{f_{\pi^N,g^N,k^N}(R)}(R) = \Omega. \tag{6}$$

Corollary 2 *An assignment rule φ satisfies Pareto efficiency and population-monotonicity if and only if there exist $\pi_+, \pi_- \in \Pi^{\mathbb{P}}$ and $\mathscr{G} = \{g^N \in G^N\}_{N \in \mathscr{P}}$ such that for all $N \in \mathscr{P}$ and all $R \in \mathscr{R}^N_{\{0,\Omega\}}$, φ assigns Ω to agent $f_{\pi_+,\mathscr{G},\pi_-}(R)$, i.e.,*

$$\varphi_{f_{\pi_+,\mathscr{G},\pi_-}(R)}(R) = \Omega. \tag{7}$$

The rule $\varphi^{\min}$ as described in Example 1 is an example of an assignment rule that satisfies all properties stated in Corollaries 1 and 2. Example 2 shows the independence of *Pareto efficiency* from *separability*, *non-bossiness*, and *population-monotonicity*. However, in order to prove the independence of *separability*, *non-bossiness*, and *population-monotonicity* from *Pareto efficiency* we need a new example.

Example 7 The following rule φ satisfies *Pareto efficiency*, but not *separability*, not *non-bossiness*, and not *population-monotonicity*. Without loss of generality, we define $\varphi(R)$ for $N = \{1, 2, 3\}$. Let $R \in \mathscr{R}^N_{\{0,\Omega\}}$. If $N_\Omega(R) \neq \emptyset$ and $|N_\Omega(R)| > 1$, then

$$\begin{aligned} \varphi_1(R) &= \Omega \quad \text{if } \Omega\, P_1\, 0, \\ \varphi_2(R) &= \Omega \quad \text{if } 0\, I_1\, \Omega, \\ \varphi_3(R) &= \Omega \quad \text{if } 0\, P_1\, \Omega, \end{aligned} \tag{8}$$

and $\varphi(R) = \varphi^{\min}(R)$ otherwise. ◊

Remark 2 Pápai (1998) considers the problem of allocating an indivisible commodity, or object, where agents are not indifferent between the alternatives "receiving nothing" and "receiving the object" and where free disposal is allowed. For this model she proves that the class of rules that satisfy *Pareto efficiency*, *strategy-proofness*, and *non-bossiness* equals the class of serial dictatorships (Pápai, 1998, Proposition 2). Furthermore, Pápai (1998) studies the trade-off between the properties *Pareto efficiency*, *strategy-proofness*, *non-bossiness*, *and non-dictatorship*.

Restricting our model to the model of Pápai (1998), the class of rules described in Corollary 1 equals the serial dictatorships described in Pápai (1998). The proofs of this result are different

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